THE CALCULATION OF LOGARITHMS

By James K. Whittemore

CONTENTS

1. Introduction. The use of logarithms is so general that it seems desirable to give here a short account of the history of their invention and of the best method for their computation. To understand these explanations is necessary a comprehension of the following definitions and elementary properties of logarithms:

If $a^x = N$, where a and N are positive numbers, x is called the "logarithm of N with the base a." We write the relation in the form

$$x = \log_a N.$$

Two bases are in common use, the base ten used in numerical work, and the "natural" base, usually denoted by e, used in the calculus. The value of e is the sum of the infinite series:

$$e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$
$$= 2.71828$$

correct to five decimals. If M is any positive number,

$$\log_a N + \log_a M = \log_a NM.$$

If p is any real number,

$$\log_a N^p = p \log_a N.$$

From these equations it follows that

$$\log_a N = \frac{\log_b N}{\log_b a} = \frac{1}{\log_N a},$$

where b is any positive number. If then the logarithms with one base b, of all positive numbers are known, the logarithms of all positive numbers with any base a, may be found by division.

We shall hereafter denote $\log_{10} N$ by $\log N$, and $\log_e N$ by lN. With this notation we may write

$$\log N = \frac{lN}{l10} = \log e \cdot lN.$$

It is proved in the calculus that

$$(1) \qquad l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{if } |x| < 1,$$

where the symbol $|x|$ means the absolute or numerical value of x. From (1) it follows that, if $|x|$ be less than one,

$$(1') \qquad \log_a (1+x) = \log_a e \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \right\}.$$

The factor, $\log_a e$, is called the modulus of the system of logarithms with the base a. We shall denote by M the modulus of the system of base ten. The value of M, correct to five decimals, is

$$M = \log e = 0.43429.$$

The series (1) is not convenient for the computation of logarithms, first because it does not converge if $|x|$ is greater than one, secondly because if $|x|$ is only a little less than one a very large number of terms must be taken to obtain an approximately correct value for $l(1+x)$. But the series is of

great importance in any discussion of the properties of logarithms, and moreover serves as a basis for the deduction of other series of great use in computation. One of the most useful of such series is the following:

$$(2)\qquad \begin{aligned} ly &= \tfrac{1}{2}\, l(y+1) + \tfrac{1}{2}\, l(y-1) \\ &\quad + \frac{1}{2y^2-1} + \frac{1}{3}\left(\frac{1}{2y^2-1}\right)^3 + \frac{1}{5}\left(\frac{1}{2y^2-1}\right)^5 + \cdots. \end{aligned}$$

This series converges for all values of y greater than one, and converges very rapidly for large values.

It may be deduced from (1) as follows: Replacing x by $-x$ in (1),

$$l(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \qquad |x| < 1.$$

Subtracting this series from (1) and dividing by two,

$$\frac{1}{2}\, l\, \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots.$$

Let x be written

$$x = \frac{1}{2y^2-1}.$$

Then in order that $|x|$ be less than one it is both necessary and sufficient that y^2 be greater than one. We find

$$\begin{aligned} \frac{1}{2}\, l\, \frac{1+x}{1-x} = \frac{1}{2}\, l\, \frac{y^2}{y^2-1} &= ly - \frac{1}{2}\, l(y+1) - \frac{1}{2}\, l(y-1) \\ &= \frac{1}{2y^2-1} + \frac{1}{3}\left(\frac{1}{2y^2-1}\right)^3 + \frac{1}{5}\left(\frac{1}{2y^2-1}\right)^5 + \cdots, \end{aligned}$$

from which series (2) is at once obtained.

For logarithms with the base ten, we have

$$(2')\qquad \begin{aligned} \log y &= \tfrac{1}{2} \log\,(y+1) + \tfrac{1}{2} \log\,(y-1) \\ &\quad + M\left\{\frac{1}{2y^2-1} + \frac{1}{3}\left(\frac{1}{2y^2-1}\right)^3 + \frac{1}{5}\left(\frac{1}{2y^2-1}\right)^5 + \cdots\right\}. \end{aligned}$$

If y is a prime integer greater than two, the numbers $y+1$ and $y-1$ are both even, and their logarithms may be expressed in terms of the logarithms of two and of integers less than y.

2. Historical. A detailed and interesting account of the invention of logarithms and of the methods of the first computers is given in the article on "logarithms" by J. W. L. Glaisher in the Encyclopedia Brittanica. From this I have taken most of the historical matter here given, and to this I refer the reader who desires a fuller account than is contained in this article.

Logarithms were invented, it is generally admitted, about 1614 by an Englishman, John Napier, Baron of Merchiston. His first published work was seen by Henry Briggs, at that time Professor of Geometry at Gresham College, London, later a professor at Oxford. Briggs visited Napier and worked with him towards the perfection of the theory, and subsequently devoted much time to the calculation of logarithms. From these two men the system of logarithms with the base ten has derived the name of Naperian or Briggsian logarithms. The word "logarithm," it may here be explained, comes from the Greek words, λόγων ἀριθμός, meaning the number of ratios, for logarithms were first regarded as a number of ratios. Thus if ten be regarded as the product of 10000 equal ratios, a, so that

$$a^{10000} = 10,$$

we may, since we find that approximately

$$a^{3010} = 2,$$

say that 0.3010 is the logarithm of two. Briggs was an ardent computer, and from 1614 to 1617 he calculated to fourteen decimals the logarithms of the integers from 1 to 1000. In 1624 he published his "Arithmetica Logarithmica" containing the logarithms to fourteen decimals of the integers from 1 to 20,000, and of those from 90,000 to 101,000. This earliest of all logarithmic tables contains all the logarithms necessary for our modern five- and six-place tables. Briggs was still occupied in calculating the logarithms of the integers between 20,000 and 90,000, when in 1628 Adrian Vlacq of Holland published a table giving to ten decimals the logarithms of all integers from 1 to 100,000. These tables have been the basis of compilation of nearly all tables of logarithms published since that time. Naturally they contained some errors.

It is noteworthy that these tables differ in arrangement from tables now in common use in that the logarithms are given to ten and fourteen decimals,

while the numbers, "arguments" of the table, contain only five figures. Modern four and five-place tables generally give the logarithm with only one more figure than is contained in the argument. That the latter arrangement is that best suited to practical numerical work appears from the following considerations: in such a table the difference of successive logarithms, that is, logarithms of arguments differing by one, is always less than forty-five units in the last decimal place, and for half of the values of the argument, less than ten units, so that interpolation for a logarithm of a number containing the same number of figures as the logarithm in the table may be performed mentally.* Now if two numbers are to be multiplied by the use of logarithms, and if at least one of these numbers is not known certainly beyond m significant figures, the product cannot be found correctly to more than m significant figures,* and just that number of figures will be given by the use of logarithms to m decimals.† Hence for multiplication of numbers of m figures it is desirable to use logarithms to m decimals, and consequently convenient, but not always necessary, to have a table of logarithms whose arguments contain $m - 1$ figures.

One may then reasonably inquire whether tables like those of Briggs and Vlacq have, in the last figures of the logarithms, any value. To this may be answered first that these tables were doubtless intended to serve as a basis for the subsequent calculation of larger tables; secondly that by interpolation the values of logarithms of intermediate numbers may be found from these tables to a large number of figures; indeed, as I shall prove,† the value of the logarithm to m decimals of any number may be found from an m-place table (one in which logarithms are given to m decimals) by interpolation with first differences if the argument is given with $\frac{1}{2}m + 1$ figures.‡

Briggs' method of calculating logarithms was extremely laborious, for at that time no developments in infinite series of the logarithm had been discovered. He extracted successive square roots of ten fifty-four times, obtaining the result,

$$10^y = 1 + 12781\ 91493\ 20032\ 35 \times 10^{-32} = 1 + a,$$

where $$y = \frac{1}{2^{54}}.$$

* See Note A.

† See Note B.

‡ For a more exact statement, see Note B.

The right hand member of this equation may be written as one plus a decimal of thirty-two figures, fifteen ciphers preceeding the first significant figure of the fractional part. This result amounts to the statement that the Briggsian logarithm of the second member of this equation is

$$1/2^{54} = 55511\ 15123\ 12578\ 2702 \times 10^{-35}.$$

Briggs discovered that the logarithms of numbers of the form $1 + x$, where x is a decimal beginning with fifteen ciphers, are very nearly proportional to the decimal x. This ratio

$$\frac{\log (1 + x)}{x} = M,$$

approximately, for a small value of x, as we see from the series (1′). He obtained the value of the ratio by dividing $1/2^{54}$ by ten raised to that power, and obtained a result to eighteen decimals of which the first sixteen agree with the correct value of M. His process was to extract successive square roots of the number whose logarithm was sought, until a root of the form $1 + x$ was obtained. The logarithm of this root was then found by multiplying x by the ratio

$$\frac{\log (1 + a)}{a},$$

and from the logarithm of the root was found at once the logarithm of the number sought.*

Much interest was taken by mathematicians of the seventeenth and eighteenth centuries in the calculation of logarithms. Most computers used methods of an arithmetical nature similar to that invented by Briggs. But as the calculus developed, less painful methods based upon the use of infinite series were invented and used by some scholars, among whom was Newton. There have been few tables made from new calculations since the publication of Vlacq's work, though many writers have computed the logarithms of some numbers. Indeed a new computation of logarithms already known cannot be regarded as a very useful service to the mathematical community. The most important calculations, after those of Briggs and Vlacq, were made by two Englishmen, Sang and Thomson, and by direction of the French government for the "Tables du cadastre." Sang published in 1871 a seven-place table of the

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* See Note C.

logarithms of the integers from 20,000 to 200,000, of which the last 100,000 were freshly computed by him. Thomson calculated to twelve figures the logarithms of all integers to 120,000. His work has been used to verify errors discovered in Vlacq's tables by a comparison with the "Tables du cadastre." An account of the latter tables may be of interest. A full description of them with explanations of the methods of their compilers is given by Lefort in an article in the fourth volume of the *Annales de l'Observatoire de Paris.*

In 1784 it was voted by the French authorities that new tables of the logarithms of numbers, of the trigonometric functions, and their logarithms should be calculated to correspond to the decimal division of the quadrant. The manuscripts of these tables, which have never been published, give to fourteen decimals the logarithms of the integers from 1 to 200,000, the natural sines, and the logarithms of sines and tangents. The intention was to publish them as twelve-place tables, but even the twelfth figure is not reliable. The work was done under the direction of an engineer, Prony. His subordinates were divided into three sections: first, five or six mathematicians, including Legendre, who were occupied with the preparation of formulas and with other purely analytical work; second, a group of seven or eight men who had at least enough mathematical knowledge to translate general formulas into numbers, and who did most of the calculating done from series; finally, a group of seventy or more computers who were occupied chiefly with the work of interpolation. The work was performed wholly in duplicate by independent workers, and required over two years for its accomplishment. The two manuscripts are deposited in Paris, one at the Observatory, the other at the Institute. The tables received, as has been stated, the name of the "Tables du cadastre." The exact meaning of the word, "cadastre" is not clear. In modern French usage it means a public register of the ownership of real property. The term is applied to other registers, especially to those carried out in great detail, and it is presumably in this sense that it was applied to these mathematical tables. The chief interest of these tables lies in the facts that no other computation on so great a scale has ever been carried through, and that this is by far the greatest work of interpolation ever undertaken.

The computation of the tables proceeded as follows. The logarithms of all integers from 1 to 10000 were computed to nineteen decimals, those of prime numbers from the series $(2')$, those of composite numbers by adding the logarithms of the factors. Thus was known the logarithm of every hundred from 1 to 1,000,000, and the logarithms of all integers between these

hundreds were obtained by interpolation. Only alternate logarithms were used as a basis for the interpolation, the others serving to check this work. The interpolation was not carried on by differences of known values, but the differences were themselves computed from series. The interpolation formulas used are of interest.

Let u_0, u_1, u_2, etc., represent successive values of the function considered. Thus,

$$u_0 = \log n, \qquad u_1 = \log(n+1), \text{ etc.}$$

We represent the first differences by Δu_0, Δu_1, etc., so that

$$u_1 = u_0 + \Delta u_1, \qquad u_2 = u_1 + \Delta u_1.$$

If the second differences are $\Delta^2 u_0$, $\Delta^2 u_1$ etc.,

$$\Delta u_1 = \Delta u_0 + \Delta^2 u_0, \qquad \Delta u_2 = \Delta u_1 + \Delta^2 u_1.$$

In general, if Δ^m be an mth difference

$$\Delta^{m-1} u_p = \Delta^{m-1} u_{p-1} + \Delta^m u_p.$$

By a combination of these formulas we find

$$u_p = u_0 + p\,\Delta u_0 + \frac{p(p-1)}{2}\,\Delta^2 u_0 + \frac{p(p-1)(p-2)}{3\,!}\,\Delta^3 u_0 + \cdots$$

This formula is called by Lefort "Mouton's Interpolation Formula," but is an immediate consequence of Newton's formula.*

In this formula the differences may be calculated from series, for

$$\Delta u_0 = \log(n+1) - \log n = \log\left(1 + \frac{1}{n}\right)$$
$$= M\left(\frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5} - \cdots\right),$$
$$\Delta^2 u_0 = \Delta u_1 - \Delta u_0$$
$$= M\left(\frac{1}{n+1} - \frac{1}{3(n+1)^3} + \cdots\right) - M\left(\frac{1}{n} - \frac{1}{3n^3} + \cdots\right)$$
$$= -M\left(\frac{1}{n^2} - \frac{2}{n^3} + \cdots\right).$$

* Markoff, *Differenzenrechnung*, p. 15.

In this way may be obtained series for the differences of all orders. In the construction of the "Tables du cadastre" differences up to and including those of the sixth order were used. The tables, if completed, would have given the logarithms to twelve figures of all arguments with six figures, but as stated, the work was not carried beyond the finding of the logarithms of numbers up to 200,000.

3. Practical Computation. If a complete table of logarithms is to be calculated, there is doubtless no better method than that of interpolation, used in the Tables du cadastre. But if a single logarithm is to be calculated to a large number of decimals, the best method is one published by Weddle in 1845 in *The Mathematician*, a method which is not directly based on infinite series. Before examining this method it is interesting to know why it should ever be necessary to compute a logarithm to a large number of decimals. This will be made clear by the statement of Gernerth in the preface to his excellent five-place tables.* Every logarithm is given by him correct to five decimals, and moreover it is indicated whether this value is larger or smaller than the correct value. His table was compiled from Vega's Thesaurus, published in Leipzig in 1794, which gives logarithms to ten decimals. But Vega's last decimal place is sometimes wrong, and Gernerth decided to consider this tenth figure as unreliable. Finding in the Thesaurus

$$\log 5.873 = 0.768\ 8600\ 008,$$

Gernerth, fearing that the last figure might be nine units too large, was in doubt whether to write

$$\log 5.873 = 0.76886 +$$

or

$$\log 5.873 = 0.76886 -,$$

and was consequently obliged to compute the logarithm to ten places. It may be imagined that in some cases it would be necessary to compute a logarithm to even more than ten figures.

Again, for a single calculation, the easiest method may sometimes be to compute the logarithms needed for the work and finally to compute the number from the resulting logarithm. Thus, if it were required to find to thirty decimals the value of $1/23^{50}$, and only Gernerth's five-place tables were at our

* A. Gernerth, *Fünfstellige gemeine Logarithmen*, Vienna, 1901.

disposal, the best method would be to calculate to fifteen decimals log 23, and then from log $1/23^{50}$, to compute the number required.

In most practical work it never becomes necessary to compute a logarithm. It is seldom necessary to use logarithms to more than seven places, and generally four or five places suffice.

A description of Weddle's method, with an example of its application, is given by Gernerth, together with the small tables needed. This method I shall now describe and illustrate.

We suppose, as we may without loss of generality, the number N, whose logarithm is sought, to be between one and ten, and write

$$N = a + d,$$

where a is an integer less than ten, and d a decimal fraction.

We now divide $a + d$ by $a + 1$, writing the quotient, which is less than one, as a decimal carried to a number of places sufficient to insure the degree of accuracy desired.* This is next multiplied by $1 + c_r/10^r$, where c_r is the difference between nine and the first figure of the quotient not nine, which figure we suppose to be in the rth decimal place. Generally we should have $r = 1$. This product will have all figures, before the rth, nines and generally the rth also. This process of multiplication by factors of the form, $1 + c_r/10^r$, may be repeated until the final product is as near one as is desired.

Then, with as small an error as is desired

$$N = (a + 1) \div \left(1 + \frac{c_1}{10}\right)\left(1 + \frac{c_2}{10^2}\right)\cdots,$$

$$\log N = \log (a + 1) - \sum \log \left(1 + \frac{c_r}{10^r}\right).$$

To follow this method it is necessary to have a table giving the logarithms of integers from one to ten, and the logarithms of numbers of the form, $1 + c_r/10^r$, c_r being always an integer less than ten. Such a table, giving the logarithms to fifteen places for all values of r up to and including sixteen, is given by Gernerth on a single page.† It remains to explain how this short table may be computed.

* In the example, seventeen; see Note D.

† Loc. cit., p. 119.

Logarithms of numbers in the form $1 + \frac{c_r}{10^r}$ may be easily found from the series (1′), when the value of M is known. The logarithms of the prime digits 2, 3, 5, and 7, and the value of M are ingeniously found by J. C. Adams in the following manner:

Let
$$a = l\,\frac{10}{9} = -\,l\left(1 - \frac{1}{10}\right),$$
$$b = l\,\frac{25}{24} = -\,l\left(1 - \frac{4}{100}\right),$$
$$c = l\,\frac{81}{80} = \quad l\left(1 + \frac{1}{80}\right),$$
$$d = l\,\frac{50}{49} = -\,l\left(1 - \frac{2}{100}\right),$$
$$e = l\,\frac{126}{125} = \quad l\left(1 + \frac{8}{1000}\right).$$

The values of a, b, c, d, and e are easily obtained from the series (1). Then we shall have
$$l\,2 = 7a - 2b + 3c,$$
$$l\,3 = 11a - 3b + 5c,$$
$$l\,5 = 16a - 4b + 7c,$$
$$l\,7 = \tfrac{1}{2}(39a - 10b + 17c - d) = 19a - 4b + 8c + e.$$

The first of these formulas follows from the equation, easily verified,
$$2 = \left(\frac{10}{9}\right)^7\left(\frac{24}{25}\right)^2\left(\frac{81}{80}\right)^3.$$

The others may be similarly proved. From the two expressions for $l\,7$ we obtain a check formula
$$a - 2b + c = d + 2e.$$

We have further
$$\frac{1}{M} = l\,10 = l\,2 + l\,5 = 23a - 6b + 10c.$$

Adams used these formulas to find to 260 decimals the natural logarithms of 2, 3, 5, and 7, and the value of M to 282 decimals. It may be noticed that the discovery of such relations as those used by Adams would be a real application of simple indeterminate equations required to be solved in integers.

We proceed to calculate by Weddle's method the logarithm of 5873, using the short tables mentioned given by Gernerth. Since Gernerth's tables are carried to fifteen places, we shall proceed with the multiplication until the first eight figures of the product are nines, when the remaining seven factors necessary for the computation may be found by inspection. The work of division and multiplication must be carried to seventeen figures to insure the correctness of the fifteenth factor. To indicate multiplication by a factor $1 + c_r/10^r$, we place this number to the right of the multiplicand, write underneath the significant figures of the product by $c_r/10^r$, correct to seventeen decimals, and, adding this product to the multiplicand, obtain the product by $1 + c_r/10^r$. The factors obtained by inspection we shall denote by $1 + a_r/10^r$. We have

$$\log 5873 = 3 + \log 5.873$$

$$\frac{5.873}{6} = \begin{array}{rl}
0.97883\ 33333\ 33333\ 33 & \qquad 1+\dfrac{2}{10^2} \\
1957\ 66666\ 66666\ 67 & \\ \hline
0.99841\ 00000\ 00000\ 00 & \qquad 1+\dfrac{1}{10^3} \\
99\ 84100\ 00000\ 00 & \\ \hline
0.99940\ 84100\ 00000\ 00 & \qquad 1+\dfrac{5}{10^4} \\
49\ 97042\ 05000\ 00 & \\ \hline
0.99990\ 81142\ 05000\ 00 & \qquad 1+\dfrac{9}{10^5} \\
8\ 99917\ 30278\ 45 & \\ \hline
0.99999\ 81059\ 35278\ 45 & \qquad 1+\dfrac{1}{10^6} \\
9999\ 98105\ 94 & \\ \hline
0.99999\ 91059\ 33384\ 39 & \qquad 1+\dfrac{8}{10^7} \\
7999\ 99284\ 75 & \\ \hline
0.99999\ 99059\ 32669\ 14 & \qquad 1+\dfrac{9}{10^8} \\
899\ 99991\ 48 & \\ \hline
0.99999\ 99959\ 32660\ 62 &
\end{array}$$

Then $c_1 = 0$, $c_2 = 2$, $c_3 = 1$, $c_4 = 5$, $c_5 = 9$, $c_6 = 1$, $c_7 = 8$, $c_8 = 9$. The remaining seven factors, $1 + a_r/10^r$, may be obtained by taking the differences

between nine and the successive figures to the fifteenth. The fifteenth figure is here correct because the errors of the approximate work cannot be greater than five in the seventeenth place.* Then

$$a_8 = 9,\ a_9 = 4,\ a_{10} = 0,\ a_{11} = 6,\ a_{12} = 7,\ a_{13} = 3,\ a_{14} = 3,\ a_{15} = 9.$$

There are two methods of using Gernerth's short table, one taking the logarithms as they are given, the other adding or subtracting to each logarithm, according as it is too small or too large, one fourth of a unit in the fifteenth place. The plus and minus signs written after the logarithms and the additional plus sign are used in the second method. The logarithms of the thirteen factors different from one are written in order, each correct to fifteen decimals.

$$\begin{array}{r l}
0.00860\ 01717\ 61918 & - \ (25\cdot 10^{-17}) \\
0.00043\ 40774\ 79319 & - \\
0.00021\ 70929\ 72230 & + \\
0.00003\ 90847\ 44584 & + \\
04342\ 94265 & - \\
03474\ 35447 & - \\
00390\ 86502 & - \\
00017\ 37178 & - \\
26058 & - \\
03040 & + \\
00130 & + \\
00013 & + \\
4 & - \\
 & + \\
\hline
0.00929\ 12495\ 40788 & - \ (50\cdot 10^{-17})
\end{array}$$

Now, $\log 6 = 0.77815\ 12503\ 83644 -$

Subtracting,

$$\log 6 - \sum_{r=1}^{8} \log\left(1 + \frac{c_r}{10^r}\right) - \sum_{r=9}^{15} \log\left(1 + \frac{a_r}{10^r}\right)$$

$$= \log 5.873 = 0.76886\ 00008\ 42856.$$

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* Note D.

This value is liable to an error less than 7.55 units in the fifteenth decimal place if Gernerth's table be used by the first method; to an error less than 3.85 units in the fifteenth decimal place if the second method is used. It is only a coincidence that the two methods give the same result. The second method enables us to say that this value is correct to fourteen decimals, but not that the value of this correct to fourteen decimals,

$$0.76886\ 00008\ 4286,$$

is the true value correct to fourteen decimals. Either method enables us to say that the true value correct to fourteen decimals is either that just given or differs from that only in having the fourteenth figure a five.

4. The Limits of Error in Several Approximations. In preparing this sketch of the history of the development of logarithms and of the methods of their computation, I have frequently had occasion to question to what extent certain processes employed can be relied upon to give correct results. In attempting to answer the questions raised I have learned first, that we can in all cases assign to the error committed in the approximation a limit which cannot be surpassed, which I shall speak of as the limit of error; second, that these limits are often smaller than I had before supposed; and thirdly, in seeking to make the limit of error as small as possible, I found myself led in Weddle's method of computing a logarithm to very definite rules for computation. These results have interested me so much that I venture to hope that others may also find them interesting and not unprofitable. For a knowledge of the limit of error in computation will not only give to the computer more confidence in his result, but will also, it seems to me, give him a much more truly scientific habit of mind. I have divided these studies into four notes referred to in the previous pages. A few preliminaries will be useful.

It is customary in giving the approximate value x', of a number x, to give x' to a stated number of decimals m, so that $|x - x'|$ is as small as possible. Then x' is said to be the value of x "correct to m decimals." Clearly we shall always have

$$|x - x'| \leqq \frac{1}{2 \cdot 10^m}.$$

If x and y are any two numbers such that

$$|x - y| \leqq \frac{1}{2 \cdot 10^m},$$

we shall find it convenient to say that y is equal to x correct to m decimals. But we must note that if x' and y' are the values of x and y correct to m decimals, we may have

$$|x' - y'| = \frac{1}{10^m}.$$

A few examples will make these points clearer. Let $x = 0.43429$; then if m is three, $x' = 0.434$; if m is four $x' = 0.4343$; if $y = 0.4346$, y is equal to x correct to three decimals, but for m equal three, $y' = 0.435$ and $y' - x' = 1/10^3$.

Finally if x and y are two numbers such that

$$|x - y| \leqq \frac{1}{10^m}$$

then also

$$|x' - y'| \leqq \frac{1}{10^m}.$$

For example, taking again m as three, and $x = 0.43429$, if $|x - y| \leqq 1/10^3$, y lies between, or is equal to one of the values

$$0.43329, \qquad 0.43529,$$

and y' is equal to one of the values

$$0.433, \qquad 0.434, \qquad 0.435,$$

no one of which differs from x' by more than $1/10^3$.

We shall have occasion to use the developments for small values of x

$$\left.\begin{aligned} \log(1 + x) &= M\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots\right\}, \\ -\log(1 - x) &= M\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \cdots\right\}. \end{aligned}\right\} \quad 0 < x < 1,$$

From these developments we may infer that, for positive values of x less than one,

$$0 < \log(1 + x) < Mx,$$

$$0 < -\log(1 - x) < \frac{Mx}{1 - x}.$$

Note A. 1. If two numbers, of which one is not known beyond m significant figures are multiplied, the product cannot be known beyond m significant figures.

2. In an m-place logarithm table whose arguments contain $m-1$ figures successive tabulated values differ by not more than forty-five units in the last decimal place, and for one half of the arguments by less than ten units.

Proof of 1.

We may, without loss of generality, suppose both of the numbers to lie between 1 and 0.1, so that each number has its first significant figure in the first decimal place. If the two numbers are x and y, and they are known to m and n figures respectively, and if the approximately correct values be x' and y', then

$$x' = x + \epsilon, \quad y' = y + \eta,$$

where

$$|\epsilon| \leqq \frac{1}{2 \cdot 10^m}, \qquad |\eta| \leqq \frac{1}{2 \cdot 10^n}, \qquad n \geqq m.$$

Then is

$$x'y' = xy + \epsilon y + \eta x + \epsilon\eta,$$

$$|x'y' - xy| = |\epsilon y + \eta x + \epsilon\eta|.$$

Since nothing is known of the signs of ϵ and η, we cannot know that the second member of the last equation is less than $|\epsilon y|$, which itself cannot be known to be less than or equal to $1/(2 \cdot 10^{m+1})$, since y is greater than 0.1; hence it cannot be known that the value of $x'y'$ is that of xy correct to more than m figures. It may also be shown that the upper limit of $|x'y' - xy|$ is $(5.75)/10^m$, but that in more than two thirds of all possible cases this difference is less than $1/10^m$.

Proof of 2.

Let a and $a+1$ be two successive values of the argument, and let the tabulated values of their logarithms, each correct to m decimals be $\log' a$ and $\log' (a+1)$. Then

$$|\log a - \log' a| \leqq \frac{1}{2 \cdot 10^m}, \ |\log (a+1) - \log'(a+1)| \leqq \frac{1}{2 \cdot 10^m}, \ a \geqq 10^{m-2}.$$

The tabular difference, Δ, is

$$\Delta = \log'(a+1) - \log' a = \log (a+1) - \log a + \theta, \qquad |\theta| \leqq \frac{1}{10^m}.$$

Now is

$$\log(a+1) - \log a = \log\left(1 + \frac{1}{a}\right) < \frac{M}{a} < \frac{44}{10^m}.$$

Then

$$\Delta < \frac{44}{10^m} + |\theta| < \frac{45}{10^m}.$$

If the first digit of a is five or greater, $\frac{M}{a}$ is less than $\frac{44}{5 \cdot 10^m}$, and

$$\Delta < \frac{8.8}{10^m} + |\theta|$$

$$< \frac{9}{10^m}.$$

Note B. The Limit of Error in Interpolation by First Differences in a Table of Logarithms. Let us consider a table giving the logarithms correct to m decimals of all integers between 10^n and 10^{n+1}, a table then with arguments of $n + 1$ figures. Let a be any value of the argument, and let $\log' a$ be the tabulated value of its logarithm. The formula for interpolation by first differences is

$$(3) \qquad \log(a + x) = \log' a + x\,[\log'(a+1) - \log' a],$$

where x is any positive number less than one. We proceed to investigate the limit of error of this formula. Let us write

$$(4) \qquad \log(a+x) = \log a + (x + \epsilon)[\log(a+1) - \log a].$$

We have from (1)

$$\log(a+x) - \log a = Ml\left(1 + \frac{x}{a}\right)$$

$$= M\left[\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} \cdots\right],$$

$$\log(a+1) - \log a = M\left[\frac{1}{a} - \frac{1}{2a^2} + \frac{1}{3a^3} \cdots\right],$$

whence, from (4), cancelling $\frac{M}{a}$,

$$\epsilon\left[1 - \frac{1}{2a} + \frac{1}{3a^2} \cdots\right] = \frac{x(1-x)}{2a} - \frac{x(1-x^2)}{3a^2} \cdots$$

$$< \frac{x(1-x)}{2a},$$

since each term of the series in the second member of the last equation is less in absolute value than the preceding.* Moreover the coefficient of ϵ in the first member is greater than $1 - \dfrac{1}{2a}$. Hence we find

$$0 < \epsilon < \frac{x(1-x)}{2a-1}.$$

The maximum value of $x(1-x)$ is $\frac{1}{4}$. Then

$$0 < \epsilon < \frac{1}{4(2a-1)}.$$

Since we know that

$$0 < \log(a+1) - \log a < \frac{M}{a},$$

we may write

$$\log(a+x) = \log a + x\,[\log(a+1) - \log a] + \epsilon',$$

where

$$0 < \epsilon' < \frac{M}{4a(2a-1)}.$$

If we write

$$\log a = \log' a + \epsilon_1, \qquad \log(a+1) = \log'(a+1) + \epsilon_2$$

we know

$$|\epsilon_1| \leqq \frac{1}{2\cdot 10^m}, \qquad |\epsilon_2| \leqq \frac{1}{2\cdot 10^m}.$$

Now

$$\log\,(a+x) = \log' a + x\,[\log'(a+1) - \log' a] + \epsilon_1(1-x) + \epsilon_2\, x + \epsilon'.$$

But

$$|\epsilon_1(1-x) + \epsilon_2\, x| \leqq |\epsilon_1|\,(1-x) + |\epsilon_2|\,x \leqq \frac{1}{2\cdot 10^m}.$$

If now ϵ' is less than $\dfrac{1}{2\cdot 10^m}$, $\log\,(a+x)$ will differ from the value obtained from (3) by less than $1/10^m$. This will be the case if

$$\frac{M}{4a(2a-1)} < \frac{1}{2\cdot 10^m},$$

---

* The ratio of the nth term to the preceding is

$$\frac{1-x^n}{1-x^{n-1}} \cdot \frac{n}{n+1} \cdot \frac{1}{a} < \frac{n^2}{n^2-1} \cdot \frac{1}{a} < 1,$$

since $x < 1$ and $a > 10$.

or if

$$(4a - 1)^2 > 4 \cdot 10^m \cdot M + 1,$$

or finally, if

$$a > \frac{\sqrt{4 \cdot 10^m \cdot M + 1} + 1}{4}.$$

If m is even it may be easily seen that this inequality is satisfied by $a \geqq 10^{\frac{m}{2}}$; if m is odd, and equal to five or greater, it is satisfied by $a \geqq 1.05 \times 10^{\frac{m-1}{2}}$. Then in a table where m is even, if the number of figures in the argument, $n + 1$, is equal to $m/2 + 1$, the logarithm of any number may be found by interpolation with the formula (3) so that the error is not more than one in the last decimal place. Thus if m be four, there are needed for this degree of accuracy arguments of three figures. If m is odd and as great as five the number of figures needed in the argument to ensure this degree of accuracy is $(m + 1)/2$ if the first figures of a are as great as 105; for values of a beginning with figures less than 105, the arguments must be given with an additional figure.*

There results from these considerations this remarkable fact: a logarithm obtained by the formula (3) from m-place tables with arguments of $m/2 + 1$ or $(m + 1)/2$ figures,† when taken correct to m decimals, cannot differ by more than one unit in the mth decimal place from the true value of the logarithm correct to m decimals. This limit of error is the same that we should have if the process of interpolation were exact. For in that case, since each tabulated

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* Gernerth, in his five-place tables, already referred to, uses arguments of four figures, with an additional table for arguments of five figures to 10800. This is, according to the theory here presented, one figure more in all cases than is necessary for reliable interpolation, but his arrangement evidently makes the interpolation much less laborious.

The seven-place tables of Dietrichkeit (Berlin, 1903) have arguments of four figures.

The eight-place tables published by direction of the French Ministry of War in 1891, and the eight-place tables of Mendizábal Tamborrel (Paris, 1891) both have arguments of five figures. According to this theory interpolation by first differences is reliable in these eight place tables, and also in the seven-place tables when a is as great as 1050.

The ten-place tables of Vega have arguments of five figures, so that interpolation by first differences is not reliable.

A brief treatment of the limit of error in a special case covered by my discussion is given by Markoff, *Differenzenrechnung*, pp. 35–39.

A rough discussion leading to an apparently similar result is given on page 63 of the tables of Dietrichkeit.

† According as m is even or odd, and when odd if $a > 1.05 \times 10^{\frac{m-1}{2}}$.

value is liable to an error of $1/(2\cdot 10^m)$, the value obtained for $\log(a+x)$ would be liable to an equal error, and consequently this value correct to m decimals might differ by one unit in the mth place from the true value correct to m decimals.*

We have, finally, supposing our m-place table to have arguments of $m/2+1$ or $(m+1)/2$ figures,† to see with what limit of error a number N, may be found from its logarithm. We may, without restriction, suppose the characteristic of $\log N$ to be n, so that N lies between two successive arguments of the table, a and $a+1$. Then a positive value of x, less than one, may be found from the equation

$$\log N = \log' a + x\,[\log'(a+1) - \log' a].$$

The determination for N is then

$$N' = a + x.$$

Now the value determined by (3) for $\log N'$ is precisely $\log N$. Hence

$$|\log N - \log N'| < \frac{1}{2\cdot 10^m} + \frac{M}{4a(2a-1)}.$$

If we write $N = A\cdot 10^{n+1}$ and $N' = A'\cdot 10^{n+1}$, A and A' both lie between 1 and 0.1, and

$$\frac{1}{M}\,|\log N - \log N'| = |lN - lN'| = l\left|\frac{N'}{N}\right| = l\left|\frac{A'}{A}\right|.$$

Let us suppose that A' is less than A. Should this not be the case the identical result would be obtained by interchanging these quantities in the following lines. We have

$$l\,\frac{A'}{A} = -\epsilon, \qquad \epsilon > 0.$$

Then is

$$A' = A\epsilon^{-\epsilon} = A\left(1 - \epsilon + \frac{\epsilon^2}{2!}\cdots\right),$$

---

* If in (3), $a+x$ is the value of a number N, correct to m figures, the limit of error in taking the second member of (3), correct to m places, for the value of $\log N$, correct to m places, is three in the last place even if a contains $m-1$ figures. But if $a+x$ is the value of N, correct to $m+1$ figures, the limit of error is, as before, one in the mth place, if $n = m/2$ when m is even, or if $n = (m-1)/2$ and $a > 1.5 \times 10^n$ when m is odd and at least equal to five. Then in logarithmic work where numbers are only approximately known it is necessary, to obtain the best results, to use these numbers correct to one more figure than is contained in the tabulated logarithms.

† According as m is even or odd, and when odd if $a > 1.05 \times 10^{\frac{m-1}{2}}$.

and

$$|A' - A| < A\epsilon < A\left\{\frac{1}{2M.10^m} + \frac{1}{4a(2a-1)}\right\}.$$

Since $\dfrac{1}{2M} < 1.152$ and $4a(2a-1) > 7a^2$, we have

$$|A' - A| < A\left\{\frac{1.152}{10^m} + \frac{1}{7a^2}\right\}.$$

Now, however n is chosen, we cannot infer that $|A' - A| \leqq 1/10^m$, so that it will be impossible to assert that the values of A and A', each correct to m decimals, differ by less than two in the last place. But if n is chosen equal to $(m-1)/2$ or $m/2$, according as m is odd or even, it is easy to see by examining the different possibilities, that $|A' - A| < 2/10^m$, and consequently that the values of A and A', each correct to m decimals, cannot differ by more than two in the last place. The truth of this statement is immediately evident when m is even. If m is odd suppose, for example, that the first digit of a is 3; then is

$$|A' - A| < 0.4\left\{\frac{1.152}{10^m} + \frac{10}{63 \cdot 10^m}\right\} < \frac{.53}{10^m}.$$

In fact it appears in this way that, if the first digit of a is seven or less, $|A' - A| < 1/10^m$. We may then say, returning to the numbers N and N', that the values of these two numbers each correct to m significant figures cannot differ by more than two in the last figure in any case, nor by more than one in the last figure if the first figure is less than eight.

Note C. The Limit of Error in Briggs' Method of Calculating a Logarithm. Briggs' method is based on the repeated extraction of square roots to a large number of decimals. It will perhaps be of interest to prove the fact, not always realized, that the square root of a number larger than one may be always found correct to at least as many decimals as the number is given, sometimes to more.

Let x' be the given value, correct to m decimals, of a number x. Then is $|x - x'| \leqq \dfrac{1}{2 \cdot 10^m}$.

We seek the square root of x by finding the square root of x'. If this process is carried out correct to n decimals, and we write the resulting number

as $\sqrt{x''}$, we have

$$|\sqrt{x'} - \sqrt{x''}| \leqq \frac{1}{2 \cdot 10^n}.$$

Now, the absolute value of the error,

$$|\sqrt{x} - \sqrt{x''}| \leqq |\sqrt{x} - \sqrt{x'}| + |\sqrt{x'} - \sqrt{x''}|$$

$$\leqq \frac{1}{2 \cdot 10^m} \cdot \frac{1}{\sqrt{x} + \sqrt{x'}} + \frac{1}{2 \cdot 10^n},$$

since

$$|\sqrt{x} - \sqrt{x'}| = \left|\frac{x - x'}{\sqrt{x} + \sqrt{x'}}\right| \leqq \frac{1}{2 \cdot 10^m} \cdot \frac{1}{\sqrt{x} + \sqrt{x'}}.$$

Then if x is greater than one, we have

$$\sqrt{x} + \sqrt{x'} > 2, \qquad \text{and if } n = m + 1,$$

$$|\sqrt{x} - \sqrt{x''}| < \frac{1}{2 \cdot 10^m}.$$

But the values of $\sqrt{x}$ and $\sqrt{x''}$, each correct to m decimals, may differ by one in the mth decimal place. We may notice that if $x > 10^{2p}$, we may, if we choose $n = m + p + 1$, assert

that
$$|\sqrt{x} - \sqrt{x''}| < \frac{1}{2 \cdot 10^{m+p}},$$

so that the values of $\sqrt{x}$ and $\sqrt{x''}$, each correct to $m + p$ decimals, differ by not more than one in the last decimal place.

It has been explained that Briggs' method of computing the logarithm of a number, N, consisted in finding a root

$$\sqrt[p]{N} = 1 + x,$$

where p is of the form 2^h, and x a decimal beginning with at least fifteen ciphers, then in writing

$$\log(1 + x) = \frac{x}{a} \log(1 + a),$$

where $\quad 1 + a = 10^y, \quad \text{where} \quad y = 1/2^{54}$

so that $\quad a = 12781\ 91493\ 20032\ 35 \times 10^{-32}.$

Let us write $x = na$; since x is to begin with fifteen ciphers, n must be less than eight. Briggs' formula becomes

$$\log(1 + na) = n \log(1 + a) = \log(1 + a)^n.$$

By Taylor's Theorem, we have

$$(1 + a)^n = 1 + na + \frac{n(n-1)a^2}{2}(1 + \theta a)^{n-2}, \qquad 1 > \theta > 0,$$

whence

$$1 + na = (1 + a)^n (1 - \epsilon),$$

where
$$(1 + a)^n \epsilon = \frac{n(n-1)a^2}{2}(1 + \theta a)^{n-2}.$$

Then
$$\log(1 + na) = n \log(1 + a) + \log(1 - \epsilon).$$

Now
$$|\log(1 - \epsilon)| < \frac{M\epsilon}{1 - \epsilon},$$

and since n is less than eight, and $\dfrac{(1 + \theta a)^{n-2}}{(1 + a)^n} < 1$, we have $\epsilon < 28a^2$.

Substituting the value of a, we find $\epsilon < 46 \times 10^{-32}$, whence

$$|\log(1 - \epsilon)| < 2 \cdot 10^{-31}.$$

Then for all values of n less then eight,

$$|\log(1 + na) - n \log(1 + a)| < 2 \cdot 10^{-31}.$$

Now we may suppose that N is always between one and ten; then to find a root of the form $1 + x$, it will never be necessary to take h greater than 54.

Now is
$$\log N = p \log(1 + x) = 2^h \log(1 + x),$$

and the error in $\log N$, computed by Briggs' method, will not be greater than

$$2^{54} \times 2 \cdot 10^{-31} < \frac{1}{2 \cdot 10^{14}}.$$

Hence the inaccuracy due to the method will not prevent the result from being correct to fourteen decimals.

Note D. The Limit of Error in Weddle's Method of Calculating a Logarithm. We suppose that the number N, whose logarithm is to be calculated, lies between one and ten. We write $N = a + d$, where a is an integer less than ten, and d a decimal fraction. We divide $a + d$ by $a + 1$, unless $a + d$ begins with the figures 1.0, when the method may be advan-

tageously modified by dividing by 1.1 instead of by two. The quotient N' is then multiplied by l factors of the form $1 + c_r/10^r$, so that the product has its first k decimal figures equal to nines. Representing the product of l factors of the form $1 + c_r/10^r$ by P_l, we have

$$N'' = N'P_l = 1 - \frac{a_{k+1}}{10^{k+1}} - \frac{a_{k+2}}{10^{k+2}} \cdots .$$

It is always possible to obtain this result where $l \leqq k + 3$, and if $l = k + 3$, we shall have $a_{k+1} = 0$. The truth of this statement appears as follows: We have always $N' > .55$; in the least favorable case multiplication by four factors of the form $1 + c_r/10^r$ will raise to nines the first two figures. When the first two or more figures in the product are nines it may be necessary to multiply by two factors to raise to nine the first figure of the product different from nine, but in the least favorable case this process will raise to nine this figure and the one next following. Generally, each multiplication raises to nine one figure of the product, so that $l = k$, but we may assert that in all cases, $l \lesseqgtr k + 3$. An example will make these explanations clearer. Suppose

$N' = 0.735$	$c_1 = 2$	
$N'P_1 = 0.8820$	$c_1' = 1$	
$N'P_2 = 0.9702$	$c_2 = 2$	
$N'P_3 = 0.98960\ 4$	$c_2' = 1$	
$N'P_4 = 0.99950\ 004$	$c_3 = 0$	$c_4 = 4$
$N'P_5 = 0.99989\ 98400\ 16$	$c_4' = 1$	
$N'P_6 = 0.99999\ 98300\ 00$		

In this rather unfavorable case if k is two, we have to take $l = 4 = k + 2$; to make the fourth decimal a nine two factors, $1 + c_4/10^4$, are used but with the fourth not only the fifth but the sixth figures become equal to nine, so that if k is six, we have $l = 6 = k$.

Having now

$$N'' = 1 - \frac{a_{k+1}}{10^{k+1}} - \frac{a_{k+2}}{10^{k+2}} \cdots,$$

we write

$$N'' \left(1 + \frac{a_{k+1}}{10^{k+1}}\right)\left(1 + \frac{a_{k+2}}{10^{k+2}}\right) \cdots = 1 - x,$$

and for x we find

$$x = \sum \frac{a_i a_j}{10^{i+j}} + \sum \frac{a_i a_j a_m}{10^{i+j+m}} \cdots,$$

where $$k+1 \leqq i \leqq j \leqq m \cdots,$$

and where not more than two indices are equal in any term of the sum. Since no value of a is greater than nine, we have

$$\sum \frac{a_i a_j}{10^{i+j}} < \frac{81}{10^{2k+2}} \sum \frac{1}{10^{\alpha+\beta}} = \frac{1}{10^{2k}},$$

where α and β take independently all positive integral values. Similarly, since m is at least equal to $k+2$,

$$\sum \frac{a_i a_j a_m}{10^{i+j+m}} < \frac{729}{10^{3k+4}} \sum \frac{1}{10^{\alpha+\beta+\gamma}} = \frac{1}{10^{3k+1}}.$$

So that, finally, we have

$$x < \frac{1}{10^{2k}}\left\{1 + \frac{1}{10^{k+1}} + \frac{1}{10^{2k+2}} \cdots\right\}$$

$$< \frac{1}{10^{2k}} \frac{1}{1 - \dfrac{1}{10^{k+1}}} < \frac{1.01}{10^{2k}}, \qquad \text{if } k > 1.$$

Now $$\log N'' + \sum_{r=k+1}^{\infty} \log\left(1 + \frac{a_r}{10^r}\right) = \log(1-x).$$

But $$|\log(1-x)| < \frac{Mx}{1-x} < \frac{1}{2.10^{2k}},$$

since x is known to be less than 1.01×10^{-2k}.

Then $$\log N'' = -\sum_{r=k+1}^{\infty} \log\left(1 + \frac{a_r}{10^r}\right) + \epsilon, \qquad |\epsilon| < \frac{1}{2\cdot 10^{2k}}$$

and $$\log N' = \log N'' + \sum_l \log\left(1 + \frac{c_r}{10^r}\right);$$

and since $\log N = \log(a+1) - \log N'$, we have

$$\log N = \log(a+1) - \sum_l \log\left(1 + \frac{c_r}{10^r}\right) - \sum_{r=k+1}^{\infty} \log\left(1 + \frac{a_r}{10^r}\right) + \epsilon.$$

If in the last sum we neglect all terms for which r is greater than m, we commit the error

$$\sum_{r=m+1}^{\infty} \log\left(1 + \frac{a_r}{10^r}\right) < M \sum_{r=m+1}^{\infty} \frac{a_r}{10^r} < \frac{M}{10^m} < \frac{1}{2\cdot 10^m}.$$

We compute $\log N$ from the formula

$$(4)\qquad \log N = \log'(a+1) - \sum_{l} \log'\left(1+\frac{c_r}{10^r}\right) - \sum_{r=k+1}^{m} \log'\left(1+\frac{a_r}{10^r}\right),$$

where $\log'$ means, as before, the tabulated value of a logarithm. If the tables used are correct to p places, each logarithm used is liable to an error of $1/(2\cdot 10^p)$, and we use

$$1 + l + m - k$$

tabulated values, or less if some values of a_r are zero. Now l is not greater than $k+3$, hence not more than $m+4$ logarithms are used. Then the limit of error in (4) is not greater than

$$E = \frac{1}{2}\left\{\frac{1}{10^{2k}} + \frac{1}{10^m} + \frac{m+4}{10^p}\right\}.$$

If the number of logarithms actually used is less than $m+4$, this number may replace $m+4$ in E.

We shall next consider what values may best be assigned to k and m, when p is given. It is desirable to make E as small as possible, and, in order to shorten the labor of computation, to take k no larger than is necessary. The least value possible for

$$\frac{1}{10^m} + \frac{m}{10^p}$$

is obtained by writing $m = p$. Then

$$\frac{1}{10^m} + \frac{m}{10^p} = \frac{p+1}{10^p}.$$

For clearly a value of $m \geqq p+1$ would give the expression a greater value. Whereas, if $m = p - h$.

$$\frac{1}{10^m} + \frac{m}{10^p} = \frac{10^h + p - h}{10^p} > \frac{p+1}{10^p}.$$

We shall then do best to put $m = p$, and

$$E = \frac{1}{2}\left\{\frac{1}{10^{2k}} + \frac{p+5}{10^p}\right\}.$$

To choose $2k$ less than p would make E unnecessarily large, and since E has inevitably a considerable value in the pth decimal place there is no gain in taking $2k$ greater than $p+1$. Then if p is even, let $k = p/2$, and

$$E = \frac{1}{2}\frac{p+6}{10^p};$$

if p is odd let $k = (p + 1)/2$, then

$$E = \frac{1}{2}\frac{p + 5.1}{10^p}.$$

In all cases if the number of logarithms actually used in (4) is less than $p + 4$, the difference may be subtracted from the numerator of E. In the example worked out in §3, we had $p = 15$, hence took $m = 15$ and $k = 8$. There were used in (4) fourteen logarithms. Hence we subtract five from the numerator of E and $E = \frac{1}{2}\frac{15.1}{10^{15}}$ and the error in the fifteenth decimal of log 5873, computed by the first method is less than 7.55. The value found, if taken correct to fourteen decimals cannot differ by more than one in the fourteenth place from the true value correct to fourteen decimals.

Now the chief part of E comes from possible inaccuracy in the tabulated logarithms. This part may be cut in half by a slight modification suggested and used by Gernerth. With each logarithm tabulated it is indicated whether the given value is smaller or larger than the true value. If the tabulated value, l', is too small, the true value, l, is between l' and $l' + 1/(2\cdot 10^{15})$, and if in place of l' we use $l' + 1/(4\cdot 10^{15})$, the difference from l cannot be more than $1/(4\cdot 10^{15})$. Similarly, if the tabulated value is too large, and in place of l' we use $l' - 1/(4\cdot 10^{15})$, the difference from l is not more than $1/(4\cdot 10^{15})$. If this modification is practised, we have

$$E' = \frac{1}{4}\left\{\frac{2}{10^{2k}} + \frac{2}{10^m} + \frac{m + 4}{10^p}\right\},$$

E' being now the limit of error in (4).

It appears that m should be given the value $p + 1$. The last logarithm in the sum, $\log'\left(1 + \frac{a_{p+1}}{10^{p+1}}\right)$, will be tabulated as zero, for

$$\log\left(1 + \frac{a_{p+1}}{10^{p+1}}\right) < \log\left(1 + \frac{1}{10^p}\right) < \frac{M}{10^p} < \frac{1}{2\cdot 10^p},$$

but in the modified calculation, its value must be written as $1/(4\cdot 10^p)$. As before, k should be chosen as $(p + 1)/2$ or $p/2$, according as p is odd or even. If p is even,

$$E' = \frac{1}{4}\frac{p + 7.2}{10^p};$$

$$\text{if } p \text{ is odd, } E' = \frac{1}{4}\frac{p + 5.4}{10^p}.$$

If the number of logarithms actually used is less than $p + 5$, the difference may be subtracted from the numerator of E'. In the example of §3, using this second method, fifteen logarithms are used, so that $E' = \frac{3.85}{10^{15}}$. But even this result does not enable us to say that the value of log 5873, computed by the second method, and taken correct to fourteen decimals is equal to the true value correct to fourteen decimals.

It remains to consider one more point. In order to determine the values of $a_k, a_{k+1}, \cdots a_p$, to how many places must the division of N by $a + 1$, and the multiplication of the quotient by the factors $1 + c_r/10^r$ be carried?

Suppose all this work to be carried out correctly to n decimals. Then the error in the value used for N' is not greater that $1/(2\cdot 10^n)$. This error is multiplied by $c_r/10^r$ and the product taken correct to n decimals, so that, in the value used for $N'(c_r/10^r)$, the limit of error is $\frac{1}{2\cdot 10^n}\left(1 + \frac{c_r}{10^r}\right)$. Then in the value found to n places for $N'\left(1 + \frac{c_r}{10^r}\right)$ the limit of error is

$$\frac{1}{2\cdot 10^n}\left\{1 + \left(1 + \frac{c_r}{10^r}\right)\right\}.$$

Repetition of the reasoning shows us that the limit of error in $N'P_l$ is

$$\frac{1}{2\cdot 10^n}\left\{1 + \left(1 + \frac{c_s}{10^s}\right) + \cdots + P_l\right\} < \frac{(l+1)P_l}{2\cdot 10^n},$$

where $1 + \frac{c_s}{10^s}$ is the last factor in P_l. We have alway P_l less than two, since $N' > .55$ and $N'P_l < 1$; generally P_l is very much less than two, and since $l < k + 3 < \frac{p+1}{2} + 3$, the error in $N'P_l$ will be less than $\frac{1}{2}\frac{p+9}{10^n}$. It will then suffice generally to choose $n = p + 2$ to have a correct value of a_p. In the example of §3, l is 7, and $P_l < \frac{10}{9}$, so that the error is less than

$$\frac{8\cdot\frac{10}{9}}{2\cdot 10^n} < \frac{4.5}{10^n}.$$

To have the fifteenth place correct it is here sufficient to take $n = 17$.

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